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REFERENCES

1. KOZLOV V.V., Topological obstacles to the integrability of natural mechanical systems. Dokl. AN SSSR, Vol.249, No.6, 1979.
2. POINCARÉ A., Selected works, Vol.1, New methods of celestial mechanics. Moscow, NUAKA, 1971.
3. SIEGEL C.L., Uber die algebraishen Integrale des restringerten Dreikörperproblems. Trans. Amer. Math. Soc., Vol.39, No.2, 1936.
4. ALEKSEEV V.M., Final motions in the problem of three bodies and symbolic dynamics. Uspekhi Matem. Nauk, Vol.36, No.4, 1981.
5. FORSTER S., The Riemann Surfaces. Moscow, MIR, 1980.
6. LEVI-CIVITA T., Sulla Regolarizzazione del problema piano dei tre corpi. Rend. Accad. Naz. Lincei. Sèr. 5, Vol.24, 1915.
7. BOLOTIN S.V., Integrability of the problem of n centres when $n>2$. Vest. MGU, Ser. Matem. No.3, 1984.
8. GAIDUKOV E.V., Asymptotic geodesics on a Riemannian manifold non-homeomorphic to a sphere. Dokl. AN SSSR, Vol.169, No.5, 1966.
9. SEIFERT C. and TREFALL V., The Variational Calculus as a Whole. Moscow, Izd. Inostr. Lit., 1947.

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ON THE STABILITY OF A SOLID ROTATING AROUND THE VERTICAL AND COLLIDING WITH A HORIZONTAL PLANE*

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The motion of a heavy solid with a convex surface above an absolutely smooth horizontal plane is considered. Collisions of the body with the plane during its motion are assumed to be absolutely elastic. The stability of such motion is investigated when the body rotates at constant angular velocity around the vertical, while its centre of mass moves between collisions on a parabola or along a fixed vertical line coinciding with the axis of rotation of the body. Stability conditions are obtained to a first approximation for arbitrary values of the parameters of the problem. Special cases of a non-rotating body with geometrical and dynamic symmetry, and of a body whose surface in the neighbourhood of the point of contact with the plane is close to spherical, are analyzed in detail. A peculiar "quantification" of stability and instability along the height of jumps of the body over the plane was found in the case of a rotating body.

The problem of the stability of the motion of a solid with a convex surface of arbitrary form and an arbitrary inertia tensor when there is a non-retaining connection, has not so far been investigated. Investigations in the theory of vibrating-collision systems have dealt with either material points or homogeneous spheres, which in the case of a smooth plane is, from the point of view of dynamics, the same.

1. Let a solid move in a gravitational field above a stationary horizontal plane. The surface of the body is assumed to be convex, and the plane is assumed to be absolutely smooth. During its motion the body may touch the plane at a point on its surface. Then, if a collision occurs, it is assumed to be absolutely elastic.

Let $Oxyz$ be a system of coordinates with origin at the point O of the horizontal plane. The Oz axis is directed vertically upward. We denote the coordinates of the centre of mass G by x, y, z, and attach to the body a system of coordinates $G\xi\eta\zeta$ whose axes are directed along its principal central axes of inertia. The orientation of the body relative to absolute space is defined by Euler's angles θ, φ, ψ which are conventionally introduced. We denote the

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point of the body surface closest to the horizontal plane $z = 0$ by M. It can be shown that the coordinates ξ, η, ζ of the point M in the system of coordinates $G\xi\eta\zeta$ are functions of θ and φ, which are determined by the form of the equation that specifies the surface of the body. The unit vector Oz in the same system of coordinates has the components

$$\gamma_1 = \sin\theta \sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi, \quad \gamma_3 = \cos\theta \tag{1.1}$$

Let m be the mass of the body, g be the free fall acceleration, and A, B, and C the moments of inertia of the body about the axes $G\xi$, $G\eta$ and $G\zeta$, respectively. The projections of the absolute angular velocity of the body on these axes are denoted by p, q, r.

For the motion of the body between collisions, when it performs a free flight above the plane, the following equations hold:

$$x^{\cdot\cdot} = 0, \quad y^{\cdot\cdot} = 0, \quad z^{\cdot\cdot} = -g \tag{1.2}$$

$$Ap^{\cdot} + (C - B)\,qr = 0, \quad Bq^{\cdot} + (A - C)\,rp = 0, \quad Cr^{\cdot} + (B - A)\,pq = 0 \tag{1.3}$$

$$p = \psi^{\cdot}\gamma_1 + \theta^{\cdot}\cos\varphi, \quad q = \psi^{\cdot}\gamma_2 - \theta^{\cdot}\sin\varphi, \quad r = \psi^{\cdot}\gamma_3 + \varphi^{\cdot} \tag{1.4}$$

To obtain the equations that define the motion of the body in time intervals which include the instant of collision of the body and plane, Eqs.(1.2)—(1.4) must be supplemented by equations that follow from the general theory of frictionless collision /1/. Denoting as usual by the minus and plus signs the kinematic characteristics of the body motion before and after the collision, we have, respectively,

$$x^{\cdot +} = x^{\cdot -}, \quad y^{\cdot +} = y^{\cdot -}, \quad z^{\cdot +} = z^{\cdot -} + \frac{1}{m} I \tag{1.5}$$

$$p^+ = p^- + \frac{\gamma_3\eta - \gamma_2\zeta}{A} I, \quad q^+ = q^- + \frac{\gamma_1\zeta - \gamma_3\xi}{B} I \tag{1.6}$$

$$r^+ = r^- + \frac{\gamma_2\xi - \gamma_1\eta}{C} I$$

$$I = -\frac{2}{k}\left[z^{\cdot -} + p^-(\gamma_3\eta - \gamma_2\zeta) + q^-(\gamma_1\zeta - \gamma_3\xi) + r^-(\gamma_2\xi - \gamma_1\eta)\right]$$

$$k = \frac{1}{m} + \frac{(\gamma_3\eta - \gamma_2\zeta)^2}{A} + \frac{(\gamma_1\zeta - \gamma_3\xi)^2}{B} + \frac{(\gamma_2\xi - \gamma_1\eta)^2}{C}$$

where I is the collision momentum, and, since the quantities $\xi, \eta, \zeta, \gamma_1, \gamma_2, \gamma_3$ do not change during the impact, they are not denoted by a minus and plus sign.

2. Suppose that at the point of interseciton of the body surface with the $G\eta$ axis for negative η, the plane tangent to the surface of the body is normal to $G\eta$. The body can execute motion in which

$$\theta = \pi/2, \quad \varphi = 0, \quad p = 0, \quad q = \psi^{\cdot} = \omega = \text{const}, \quad r = 0$$
$$x^{\cdot} = \text{const}, \quad y^{\cdot} = \text{const}, \quad z^{\cdot +} = -z^{\cdot -} = \sqrt{2gH} = \text{const}$$

In this motion the axis $G\eta$ is vertical, and the body rotates around it at constant angular velocity. As the result of collisions the body periodically jumps over the plane, and the maximum distance of the lowest point M of its surface from the plane is H. The motion is periodic of period $\tau = 2\sqrt{2H/g}$ equal to the time interval between two consecutive collisions between the body and plane. Between collisions the centre of the body mass moves either on a parabola or along a given vertical line, depending on whether the constant quantity $x^{\cdot 2} + y^{\cdot 2}$ is zero or non-zero. Below, without loss of generally, we will consider only the case when $x^{\cdot 2} + y^{\cdot 2} = 0$.

Let us investigate the stability of that motion relative to perturbations of the angles θ, φ, the projections of the angular velocity p, q, r, and of the height of the jump H. Linearization of Eqs.(1.3)—(1.5) in the neighbourhood of this motion shows that to a first approximation the height of the jump H and the angular velocity $\psi^{\cdot}$ of rotation of the body around the vertical line are constant. If we set $\theta = \pi/2 + x_1$, $\varphi = x_2$, $p = x_3$, $r = x_4$, then from (1.3) we obtain for variables x_i $(i = 1, 2, 3, 4)$ a system of equations of the first approximation

$$x_1^{\cdot} = -\omega x_2 + x_3, \quad x_2^{\cdot} = \omega x_1 + x_4 \tag{2.1}$$
$$x_3^{\cdot} = \frac{B - C}{A}\omega x_4, \quad x_4^{\cdot} = -\frac{B - A}{C}\omega x_3$$

that defines the perturbations of the motion of the body in the intervals between collisions.

From (1.6) we obtain equations that connect the kinematic characteristics of the motion of the body before and after collision to a first approximation in x_i

$$x_3^+ = x_3^- + \frac{mg\tau}{A}\left[(h - l_2)\,x_1 - l x_2\right]^- \tag{2.2}$$

$$x_4^+ = x_4^- - \frac{mg\tau}{C}[lx_1 - (h - l_1)x_2]^-$$
$$l_1 = r_1 \sin^2 \alpha + r_2 \cos^2 \alpha, \quad l_2 = r_1 \cos^2 \alpha + r_2 \sin^2 \alpha$$
$$l = (r_2 - r_1) \sin \alpha \cos \alpha$$

where H is the distance between the centre of mass of the body and the point M in unperturbed motion, r_1, r_2 are the principal radii of curvature of the surface of the body at point M, and α is the angle between the axis $G\zeta$ and the curvature line that corresponds to r_1 and is measured from the axis $G\zeta$ counterclockwise, when viewed from the side of the positive semi-axis $G\eta$ which in unperturbed motion is vertical.

3. The fundamental matrix $\mathbf{X}(t)$ of system (2.1) that satisfies the condition $\mathbf{X}(0) = \mathbf{E}$ is

$$\mathbf{X}(t) = \left\| \begin{matrix} \cos\omega t & -\sin\omega t & \frac{A}{B\omega}(\sin\omega t + \varkappa\sin\Omega t) & \frac{C}{B\omega}(\cos\omega t - \cos\Omega t) \\ \sin\omega t & \cos\omega t & -\frac{A}{B\omega}(\cos\omega t - \cos\Omega t) & \frac{C}{B\omega}\left(\sin\omega t + \frac{1}{\varkappa}\sin\Omega t\right) \\ 0 & 0 & \cos\Omega t & \frac{C}{A\varkappa}\sin\Omega t \\ 0 & 0 & -\frac{A\varkappa}{C}\sin\Omega t & \cos\Omega t \end{matrix} \right\|$$
$$\varkappa^2 = \frac{C(B-A)}{A(B-C)}, \quad \Omega^2 = \frac{(B-A)(B-C)}{AC}\omega^2$$

Let $\mathbf{x}^T = (x_1, x_2, x_3, x_4)$, where T is the symbol of transposition. Taking into account that during the collision the quantities x_1 and x_2 do not vary, we then rewrite (2.2) in the form $\mathbf{x}^+ = \mathbf{Y}\mathbf{x}^-$, where

$$\mathbf{Y} = \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{mg\tau}{A}(h - l_2) & -\frac{mg\tau}{A}l & 1 & 0 \\ -\frac{mg\tau}{C}l & \frac{mg\tau}{C}(h - l_1) & 0 & 1 \end{matrix} \right\|$$

Denoting by $\mathbf{x}^\circ$ the value of the vector $\mathbf{x}$ before the first collision, its value $\mathbf{x}^1$ prior to the second collision is calculated from the formula $\mathbf{x}^1 = \mathbf{Z}\mathbf{x}^\circ$, where $\mathbf{Z} = \mathbf{X}(\tau)\mathbf{Y}$, and $\mathbf{X}(\tau)$ is the value of the matrix $\mathbf{X}(t)$ at the instant of time equal to the period of the motion considered here. Prior to the $(k+1)$-th collision $\mathbf{x}^k = \mathbf{Z}\mathbf{x}^\circ$ $(k = 0, 1, 2, \ldots)$.

For the stability of motion it is necessary that the characteristic equation of matrix $\mathbf{Z}$ should not have roots with moduli higher than unity. When there are no such roots, but roots with moduli equal to unity are present it is necessary in the Jordan form of matrix $\mathbf{Z}$ for the cells corresponding to these roots to be of the first order.

As shown by calculations, in the problem considered here the characteristic equation will be reciprocal

$$\rho^4 - a_1\rho^3 + a_2\rho^2 - a_1\rho + 1 = 0 \tag{3.1}$$
$$a_1 = 2(\cos\omega\tau + \cos\Omega\tau) + \frac{mg\tau}{B\omega}\left[(h - l_1)\left(\sin\omega\tau + \frac{1}{\varkappa}\sin\Omega\tau\right) + \right.$$
$$\left.(h - l_2)(\sin\omega\tau + \varkappa\sin\Omega\tau)\right]$$
$$a_2 = 2 + 4\cos\omega\tau\cos\Omega\tau + \left(\frac{mg\tau}{B\omega}\right)^2\left[2(1 - \cos\omega\tau\cos\Omega\tau) + \right.$$
$$\left.\left(\varkappa + \frac{1}{\varkappa}\right)\sin\omega\tau\sin\Omega\tau\right] + 2\frac{mg\tau}{B\omega}\left[(h - l_1)\left(\sin\omega\tau\cos\Omega\tau + \right.\right.$$
$$\left.\left.\frac{1}{\varkappa}\cos\omega\tau\sin\Omega\tau\right) + (h - l_2)(\sin\omega\tau\cos\Omega\tau + \varkappa\cos\omega\tau\sin\Omega\tau)\right]$$

The region of stability is defined by the system of inequalities /2/

$$-2 < a_2 < 6, \quad 4(a_2 - 2) < a_1^2 < {}^1/_4(a_2 + 2)^2 \tag{3.2}$$

When therese inequalities are satisfied, the characteristic equation has only simple roots whose moduli are equal to unity. Outside the region defined by inequalities (3.2), Eq.(3.1) has at least one root with modulus exceeding unity.

4. The analysis of stability regions (3.2), depending on the parameters of the problem, is generally complicated. Hence we shall consider the most interesting special cases. Let $\omega = 0$, i.e. the body in unperturbed motion does not rotate around the axis $G\eta$ which performs its motion along a given vertical line. In that case for any physically possible relations between the moments of inertia A, B, C, the coefficients of Eq.(3.1) become

$$a_1 = \varkappa_1 + 4, \quad a_2 = 2\varkappa_1 + \varkappa_2 + 6$$

$$\varkappa_1 = mg\tau^2\left(\frac{h-l_1}{C} + \frac{h-l_2}{A}\right), \quad \varkappa_2 = \frac{(mg\tau^2)^2}{AC}(h-r_1)(h-r_2)$$

It can be shown that, when the three special cases: 1) $r_1 = r_2$, $A = C$, 2) $\alpha = 0$, $(h - r_1)/A = (h - r_2)/C$, and 3) $\alpha = \pi/2$, $(h - r_1)/C = (h - r_2)/A$ that conform to the boundary $a_1^2 = 4(a_2 - 2)$ of the stability region, are exluced, conditions (3.2) are equivalent to the following three inequalities:

$$h < r_1, \quad h < r_2, \quad H < H_* \tag{4.1}$$
$$H_* = [A(l_1 - h) + C(l_2 - h) - \{[A(l_1 - h) - C(l_2 - h)]^2 + 4ACl^2\}^{1/2}][4m(r_1 - h)(r_2 - h)]^{-1}$$

Hence this motion is unstable, if the centre of mass in the unperturbed motion is higher than at least one of the centres of curvature of the surface of the body at point M, or the height of the jump H of the body over the plane exceeds the critical value H_*.

For example, let the body be a homogeneous ellipsoid whose surface is defined in the system of coordinates $G\xi\eta\zeta$ by the equation $\xi^2/a^2 + \eta^2/b^2 + \zeta^2/c^2 = 1$. Then

$$h = b, \quad r_1 = \frac{c^2}{b}, \quad r_2 = \frac{a^2}{b}, \quad \alpha = 0, \quad A = \frac{m}{5}(b^2 + c^2), \quad C = \frac{m}{5}(a^2 + b^2)$$

It follows from (4.1) that when $\omega = 0$ the motion of the ellipsoid is stable, if the shortest of its semiaxes is directed along the vertical, and the height of the ellipsoid jump over the plane does not exceed $H_* = b(a^2 + b^2)/[10(a^2 - b^2)]$ when $a \geqslant c$ or $H_* = b(c^2 + b^2)/[10(c^2 - b^2)]$ when $a < c$.

Note that when m, h, A, C, r_1, r_2 are fixed, the critical height H_* of the jump of the body is a function of the angle α. It is maximum when $\alpha = 0$ of $\alpha = \pi/2$, i.e. when the lines of curvature of the body at the point M are parallel to the axes of inertia $G\zeta$ and $G\xi$ which in the unperturbed motion of the body are horizontal.

5. Let $\omega \neq 0$ and the body be dynamically and geometrically symmetric, i.e. $A = C$, $r_1 = r_2 = r$. In that case the coefficients of the characteristic equation (3.1) are

$$a_1 = 4\lambda_1\lambda_2, \quad a_2 = 4(\lambda_1^2 + \lambda_2^2) - 2$$
$$\lambda_1 = \cos\left(\frac{B-2A}{2A}\omega\tau\right), \quad \lambda_2 = \frac{mg\tau}{B\omega}(h-r)\sin\left(\frac{B\omega\tau}{2A}\right) + \cos\left(\frac{B\omega\tau}{2A}\right)$$

The stability conditions (3.2) reduce to the system of inequalities

$$0 < \lambda_1^2 + \lambda_2^2 < 2, \quad (1 - \lambda_1^2)(1 - \lambda_2^2) > 0, \quad |\lambda_1| \neq |\lambda_2|$$

which when $|\lambda_1| \neq 1$ and $|\lambda_1| \neq |\lambda_2|$ reduces to a single inequality

$$f(\tau) < 0 \tag{5.1}$$
$$f(\tau) = \left[\frac{mg(r-h)\tau}{B\omega} + \operatorname{tg}\left(\frac{B\omega\tau}{4A}\right)\right]\left[\frac{mg(r-h)\tau}{B\omega} - \operatorname{ctg}\left(\frac{B\omega\tau}{4A}\right)\right]$$

If inequality (5.1) is satisfied with a reverse sign, the motion is unstable. The regions of stability and instability are shown in Fig.1 in the plane of the parameters

$$\sigma = B\omega\tau/(4A), \qquad \delta = 4Amg(r-h)/(B\omega)^2$$

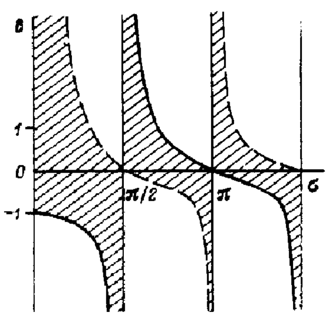

Fig.1

Since the conditions of stability are independent or of the sign of ω, σ is assumed to be a non-negative quantity. The solid lines in Fig.1 relate to the curves $\delta = -\operatorname{tg}\sigma/\sigma$ and the dashed lines to the curves $\delta = \operatorname{ctg}\sigma/\sigma$. Regions in which the inequality (5.1) is satisfied are shaded.

Some peculiar quantization appears in regions of stability and instability at the height H of the jump of the body in unperturbed motion. A denumerable set of alternating intervals of stability and instability exist, which extend infinitely upward. No matter how large the height of jump of the body, the set of intervals of stability and instability indicated here necessarily exists at a considerable height.

Let us consider in detail the results of an analysis of inequalities (5.1).

Let $r > h$, i.e in unperturbed motion the centre of mass is below the centre of the spherical centre of the surface of the body at point M by which the body collides with the plane. The instability regions are defined by the inequalities

$$\frac{g\tau_n^2}{8} < H < \frac{g}{2}\left(\frac{n\pi A}{\omega B}\right)^2 \quad (n = 1, 2, \ldots)$$

where τ_n are the roots of the equation $f(\tau) = 0$ numbered in ascending order. As n increases, the instability intervals become wider. When ω increases the regions of instability contract and concentrate close to the heights

$$H_k = gk^2\pi^2A^2/(2\omega^2B^2) \quad (k = 1, 2, \ldots)$$

Note that when $r > h$ for any ω, an instability interval exists along the height H, beginning at the plane of the jump. It is defined by the inequality

$$0 < H < H' = \frac{g\tau'^2}{8}, \quad \frac{mg(r-h)\tau'}{B\omega} = \operatorname{ctg}\left(\frac{B\omega\tau'}{4A}\right)$$

When $\omega \to 0$, we have $H' \to H_*$, where H_* is the right side of the third of inequalities (4.1) calculated for $A = C$.

Now suppose $r < h$. If the inequality $B^2\omega^2 + 4\,Amg\,(r-h) > 0$ (a condition that is similar to that of Maievskii's) is satisfied, a stability interval exists along the height which begins at the plane of the jump

$$0 < H < \frac{g}{2}\left(\frac{\pi A}{\omega B}\right)^2$$

When the Maievskii-type condition is not satisfied, an interval of instability begins at the plane of jump.

When $r < h$, the instability intervals are generally defined by the inequalities

$$\frac{g}{2}\left(\frac{n\pi A}{\omega B}\right)^2 < H < \frac{g\tau_n^2}{8}$$

with $n = 1, 2, \ldots$, if the Maievskii-type condition is satisfied, and $n = 0, 1, 2, \ldots$ otherwise.

At first sight the cause of the strange quatization of stability and instability intervals is that, when $\omega \neq 0$, the axis of symmetry of the body, in the perturbed state, performs oscillations between collisions, which results in alterations of the stability and instability region along the height of jump of the body over the plane. In Sect.4, where $\omega = 0$, the quantization of stability and instability intervals does not occur, since in the intervals between collisions, the motion of the body is not oscillatory. The perturbations of x_1, x_2 increase linearly with time, when $\omega = 0$.

6. Let the surface of the body near the point of its collision with the plane be close to the section of a sphere, and let the centre of mass be near the sphere centre. This means that the quantities h, r_1, r_2 are close to each other. We assume that they differ by quantities of order ε.

If $\varepsilon = 0$ and B is the mean moment of inertia of the body, then for the coefficients of the characteristic equation (3.1) we have the following expressions:

$$a_1 = 2(\cos\omega\tau + \operatorname{ch}\Omega_*\tau), \quad a_2 = 2 + \cos\omega\tau\operatorname{ch}\Omega_*\tau$$
$$\left(\Omega_* = \sqrt{\frac{(B-A)(C-B)}{AC}}\,\omega\right)$$

The condition of stability $a_1^2 - \frac{1}{4}(a_2+2)^2 < 0$ in (3.2) reduces to the inequality $\sin^2\omega\tau\operatorname{sh}^2\Omega_*\tau < 0$ which when $\sin\omega\tau \neq 0$ is satisfied with the opposite sign. Hence, if B is the mean moment of inertia and $\omega\tau \neq 0, \pm\pi, \pm 2\pi\ldots$, then for fairly small ε we have instability.

If B is the largest or the smallest moment of inertia, then for $\varepsilon = 0$ the coefficients of the characteristic equation (3.1) have the form

$$a_1 = 2(\cos\omega\tau + \cos\Omega\tau), \quad a_2 = 2 + 4\cos\omega\tau\cos\Omega\tau$$
$$\left(\Omega = \sqrt{\frac{(B-A)(B-C)}{AC}}\,\omega\right)$$

and the stability conditions (3.2), when $\varepsilon = 0$, are in the form of a system of inequalities

$$\sin\omega\tau\sin\Omega\tau \neq 0, \quad \sin\frac{\omega+\Omega}{2}\tau\sin\frac{\omega-\Omega}{2}\tau \neq 0, \quad \cos\omega\tau\cos\Omega\tau \neq \pm 1 \tag{6.1}$$

The first and second of these inequalities are not satisfied in the following cases:

$$1)\ \omega\tau = N_1\pi, \quad 2)\ \Omega\tau = N_2\pi, \quad 3)\ (\omega+\Omega)\tau = 2N_3\pi, \quad 4)\ (\omega-\Omega)\tau = 2N_4\pi \tag{6.2}$$

where N_i are integers. The last of inequalities (6.1) is not satisfied when the first two conditions of (6.2) are simultaneously satisfied.

When at least one of Eqs.(6.2) is satisfied, there is resonance between the oscillations frequencies ω and Ω of the body in its motion between collisions and the frequency $2\pi/\tau$ of jumps of the body; in the unperturbed motion an integral relation exists between these oscillations.

When there is no resonance, i.e. none of Eqs.(6.2) is satisfied, for fairly small $\varepsilon \neq 0$

the motion of the body is stable. If, however, there is resonance, then for small $\varepsilon \neq 0$ instability is possible. We shall consider only non-multiple resonances, assuming that when $\varepsilon = 0$ only one of Eqs.(6.2) is satisfied.

Calculations based on the stability conditions (3.2) show that in resonance cases 1), 2), and 3) when $\varepsilon \neq 0$ instability to a first approximation with respect to ε apears only if the centre of mass of the body lies between the centres of curvature of the surface of the body at point M. More exactly, in cases 1), 2), and 3) with $\varepsilon \neq 0$ suppose we have, respectively

$$\omega\tau = N_1\pi + \mu, \quad \Omega\tau = N_2\pi + \mu, \quad (\Omega + \omega)\tau = 2N_3\pi + \mu$$

Instability regions in cases 1) and 2) are defined by the inequality

$$|\mu| < \frac{mg\tau}{B\omega}\sqrt{(h - r_1)(r_2 - h)} \tag{6.3}$$

and in case 3) by the inequality

$$|\mu| < \frac{mg\tau}{B\omega}\sqrt{(h - r_1)(r_2 - h)}\,|\varkappa^{1/2} - \varkappa^{-1/2}| \tag{6.4}$$

In the resonance case 4) instability to a first approximation with respect to ε appears, if in the unperturbed motion the centre of mass of the body lies either above or below both centres of curvature. If we assume $(\omega - \Omega)\tau = 2N_4\pi + \mu$, the respective instability regions are defined by the inequality

$$|\mu| < \frac{mg\tau}{B\omega}\sqrt{(h - r_1)(h - r_2)}\,(\varkappa^{1/2} + \varkappa^{-1/2}) \tag{6.5}$$

The quantity τ in (6.3)–(6.5) is equal to its value in the corresponding resonance formula of (6.2). When $\varepsilon \neq 0$ the existence of resonances has resulted in the quantization of instability regions along the height of the jump of the body over the plane.

REFERENCES

1. LEVI-CIVITA T. and AMALDI U., Lezioni di Meccanica Razionale, Vol.2, pt.2. Bologna. Zanichelii, 1927.
2. LIAPUNOV A.M., On the Stability of Motion in the Special Case of the Three-Body Problem. Collected Works, Vol.1, Moscow, Izd. AN SSSR, 1954.

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